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An alternative approach to canonical quantization of the radiation damping

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Abstract. Inspired by some works on quantization of dissipative systems, in particular the ones treating the damped harmonic oscillator and by a paper due to Lukierski, we consider the dissipative system of a charge interacting with its own radiation, which is the origin of radiation damping. Using the indirect Lagrangian representation we obtained a Lagrangian formalism with a Chern–Simons-like term. A Hamiltonian analysis is also done in commutative and non-commutative scenarios, which leads to the quantization of the system.

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1 Introduction

The study of dissipative systems in quantum theory is of strong theoretical interest and of great relevance in practical applications. However, the standard quantization scheme, based on the existence of either a Hamiltonian or a Lagrangian function for the system in which we are interested, is not applicable when the Lagrangian or Hamiltonian has an explicit time dependence.

Among some approaches to study dissipative systems, there is one, where in order to implement a canonical quantization scheme, one must first double the phase-space dimensions, so as to deal with an effective isolated system (indirect representation) [2,5]. The new degrees of freedom thus introduced may be represented by a single equivalent (collective) degree of freedom for the bath, which absorbs the energy dissipated by the system. An important system with appropriate characteristic that allows one to use the indirect representation is the study of quantum dynamics of an accelerated charge. It is a dissipative system once an accelerated charge loses energy, linear momentum, and angular momentum carried by the radiation field [4,6]. The effect of these losses to the motion of charge is known as radiation damping [6].

The process of radiation damping is important in many areas of electron accelerator operation [7], like in recent experiments with intense-laser relativistic-electron scattering at lasers frequencies and field strengths where radiation reaction forces begin to become significant [8,9].

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The purpose of this letter is to present an alternative approach to canonical quantization of the RD based on the doubling of the degrees of freedom, where we get a Lagrangian model for the system, which is similar to one obtained by Lukierski et al. in [12] for the free particle, with a Chern–Simons-like term with high order derivatives. This letter is organized as follows. The Lagrangian and Hamiltonian formalism are derived in hyperbolic coordinates, writing down the corresponding constraints and their Dirac brackets of the independent canonical variables. The Galilean symmetry algebra has also been provided. Our model can be described either in terms of phase space variables with commuting space coordinates, or in terms of new phase space variables with non-commutative space coordinates. In this space, we have the system, which is originaly described by an equation of motion whose solution presents properties like preacceleration (the particle is accelerated before a force is applied) [6], described as two coupled harmonic oscillators with the coupling term due to the effect of the field reaction on the system. So, when the field reaction is not present, the system, in the non-commutative space, behaves like a set of two decoupled harmonic oscillators. For the choice of phase space with non-commutative space coordinates, we see that the Hamiltonian of the model describes a free motion in this space supplemented by internal degrees of freedom. After considering free motion in the non-commutative space we introduce interactions in the classical space with a potential term which depends on non-commuting $D = 2$ space coordinates. Introducing the standard quantized oscillator variables the quantum Hamiltonian has been provided in terms of the Casimir operator. We also observe that in the limit where the dissipation is not present our Hamiltonian formulation describes the dynamics of the two undamped harmonic oscillator motion.

2 The Lagrangian and Hamiltonian formalism

We begin with a review of the problem of the radiation damping. The equation of motion of one-dimensional radiation damping [6], without external force, is

$$m\ddot{x} - \gamma \dddot{x} = 0, \tag{1}$$

where $\gamma = \frac{2}{3}\frac{me^2}{c^3}$ and m are independent of time.

Since the system (1) is dissipative a straightforward Lagrangian description leading to a consistent canonical formalism is not available. To develop a canonical formalism we are required to consider (1) along with its time-reversed image [5]

$$m\ddot{y} + \gamma \dddot{y} = 0 \tag{2}$$

so that the composite system is conservative. The system (1) and (2) can be derived from the Lagrangian [10]

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}\left(\dot{x}\ddot{y} - \ddot{x}\dot{y}\right), \tag{3}$$

where x is the RD coordinate and y corresponds to the time-reversed counterpart. So, the system made of the RD and of its time-reversed image globally behaves as a closed system. Introducing the hyperbolic coordinates x_1 and x_2 [11] where

$$x = \frac{1}{\sqrt{2}}\left(x_1 + x_2\right); \quad y = \frac{1}{\sqrt{2}}\left(x_1 - x_2\right), \tag{4}$$

the above Lagrangian can be written in a compact notation as

$$L = \frac{m}{2} g_{ij}\, \dot{x}_i \dot{x}_j - \frac{\gamma}{2}\epsilon_{ij}\, \dot{x}_i \ddot{x}_j, \tag{5}$$

where the pseudo-Euclidean metric g_{ij} is given by $g_{11} = -g_{22} = 1$, $g_{12} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$. This Lagrangian is similar to the one discussed by Lukierski et al. [12] (that is a special non-relativistic limit of a relativistic model of the particle with torsion investigated in [13]), but in this case we have a pseudo-Euclidean metric. The equations of motion corresponding to the Lagrangian (5) are

$$m\ddot{x}_1 - \gamma \dddot{x}_2 = 0, \quad m\ddot{x}_2 - \gamma \dddot{x}_1 = 0. \tag{6}$$

Now, due to the presence of a second order derivative in the Lagrangian, we have to introduce two momenta

$$p_i = \frac{\partial L}{\partial \dot{x}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}_i}\ ; \ \ \tilde{p}_i = \frac{\partial L}{\partial \ddot{x}_i}. \tag{7}$$

In our case

$$p_i = m g_{ij}\dot{x}_j - \gamma\epsilon_{ij}\ddot{x}_j\ ; \ \ \tilde{p}_i = \frac{\gamma}{2}\epsilon_{ij}\dot{x}_j. \tag{8}$$

The Hamiltonian hence reads

$$\begin{aligned} H &= \dot{x}_i p_i + \ddot{x}_i \tilde{p}_i - L \\ &= \frac{m}{2} g_{ij}\dot{x}_i\dot{x}_j - \gamma\epsilon_{ij}\dot{x}_i\ddot{x}_j, \end{aligned} \tag{9}$$

or, using (8), we have

$$H = \frac{2m}{\gamma^2}\tilde{p}_i g_{ij}\tilde{p}_j - \frac{2}{\gamma} p_i \epsilon_{ij}\tilde{p}_j. \tag{10}$$

Note that this theory has two constraints,

$$\chi_i = \dot{x}_i + \frac{2}{\gamma}\epsilon_{ij}\tilde{p}_j, \tag{11}$$

where the eight-dimensional phase space is given by $(x_i, \dot{x}_i, p_i, \tilde{p}_i)$. These constraints lead to the replacement of the canonical Poisson brackets:

$$\{x_i, p_j\} = \delta_{ij} \quad \{\dot{x}_i, \tilde{p}_j\} = \delta_{ij}, \tag{12}$$

where the remaining Poisson brackets are all null, by the Dirac brackets of the independent canonical variables $(x_i, \dot{x}_i, p_i)$

$$\{x_i, p_i\}_{\mathrm{D}} = \delta_{ij}, \quad \{\dot{x}_i, \dot{x}_j\}_{\mathrm{D}} = \frac{1}{\gamma}\epsilon_{ij}, \tag{13}$$

with all Dirac brackets null. There is another combination of independent canonical variables, which is $(x_i, p_i, \tilde{p}_i)$ [12].

The Hamiltonian equations of motion take the form

$$\begin{aligned} \dot{x}_i &= \{x_i, H\}_{\mathrm{D}} = -\frac{2}{\gamma}\epsilon_{ij}\tilde{p}_j; \\ \dot{p}_i &= \{p_i, H\}_{\mathrm{D}} = 0; \\ \dot{\tilde{p}}_i &= \{\tilde{p}_i, H\}_{\mathrm{D}} = \frac{m}{\gamma}\epsilon_{in}g_{nj}\tilde{p}_j - \frac{1}{2}p_i. \end{aligned} \tag{14}$$

We can also use the Faddeev–Jackiw method [15] to obtain the brackets (13). Introducing a Lagrange multiplier which equates $\dot{x}$ to z, and replacing all differentiated x-variables in the Lagrangian (5) by z-variables, one has a first-order Lagrangian:

$$L^{(0)} = p_i\dot{x}_i - \frac{\gamma}{2}\epsilon_{ij}z_i\dot{z}_j + \frac{m}{2}g_{ij}z_i z_j - p_i z_i, \tag{15}$$

whose canonical structure can be analyzed by the Faddeev–Jackiw method, yielding the following symplectic matrix:

$$f^{(0)} = \begin{pmatrix} 0 & -\mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 \\ 0 & 0 & -\gamma\epsilon_{ij} \end{pmatrix}, \tag{16}$$

and considering that the matrix (16) is inversible, we get the Poisson brackets given by

$$\{Y_m, Y_n\}_{\mathrm{D}} = f^{-1}_{mn}. \tag{17}$$

By (16) and (17) we get the symplectic structure (13).

To obtain the quantized form of the canonical commutation relations (13) as well as the Heisenberg equations of motion we perform the replacement

$$\{\cdot,\cdot\}_{\mathrm{D}} \to (1/\mathrm{i}\hbar)[\cdot,\cdot]\ . \tag{18}$$

3 The Noether charges and their symmetries

It is important to point out that this model, described by the Lagrangian (5), presents Galileo symmetry. So, let

Table 1. The symmetries and corresponding generators

Time-tanslation	$G_\tau = H\tau$
	$H = \frac{2m}{\gamma^2}\tilde{p}_i g_{ij}\tilde{p}_j - \frac{2}{\gamma}p_i\epsilon_{ij}\tilde{p}_j$
Rotation	$G_r = J\phi$
	$J = x_i\epsilon_{ij}p_j - \frac{2}{\gamma}\tilde{p}_i^2$
Galilei boost	$G_{v_i} = B_i v_i$
	$B_i = p_i t + 2\tilde{p}_i - m g_{ij} x_j$
Space-translation	$C_t = P_i\delta_i$
	$P_i = p_i$

us consider a Lagrangian $L(x_i;\dot{x}_i;\ddot{x}_i)$ which depends on the first and second time derivatives. The variation of the action $S = \int \mathrm{d}t\ L$ under the change $x_i \to x_i + \delta x_i$ takes the form

$$\delta S = \int \delta L = \int \mathrm{d}t\ \left(\delta x_i \frac{\partial L}{\partial x_i} + \delta\dot{x}_i \frac{\partial L}{\partial \dot{x}_i} + \delta\ddot{x}_i \frac{\partial L}{\partial \ddot{x}_i}\right)$$
$$= \int \mathrm{d}t\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x_i p_i + \delta\dot{x}_i\tilde{p}_i\right), \qquad (19)$$

from which we obtain the following formulae for the generator:

$$G(t) = \delta x_i p_i + \delta\dot{x}_i\tilde{p}_i, \qquad (20)$$

which is conserved, $\frac{\mathrm{d}}{\mathrm{d}t}G(t) = 0$.

Let us list the generators of the symmetry for the Lagrangian (5) (see [12] for details) in Table 1.

Here τ is the translation shift of the time variables, δ_i is the translation shift of variables and ϕ is the rotation angle of the variables.

4 Quantization of the free model with non-commuting space coordinates

Now, in analogy with [12], our model can be described in terms of new phase space variables with non-commutative space coordinates given by the relations $[X_i, X_j] = \mathrm{i}\frac{k}{m^2}\epsilon_{ij}$ [12, 14]. That is important, because thus we can see that the dynamics in the model considered (see (10)) can be separated into two independent sectors – describing external and internal dynamics. For our Galilean system, described by Lagrangian (5), the non-commuting position variables X_i can be expressed as

$$X_i = x_i - \frac{2}{m}g_{ij}\tilde{p}_j. \qquad (21)$$

Considering $P_i = p_i$ and redefining the second pair of the momenta $\tilde{p}_i$ by

$$\tilde{P}_i = \frac{\gamma}{2m}g_{ij}p_j + \epsilon_{ij}\tilde{p}_j, \qquad (22)$$

we obtain the following standard canonical Dirac brackets for the six phase space variables $(X_i, P_i, \tilde{P}_i)$:

$$\{X_i, X_j\}_{\mathrm{D}} = -\frac{\gamma}{m^2}\epsilon_{ij}\ ,\ \{\tilde{P}_i, \tilde{P}_j\}_{\mathrm{D}} = \frac{\gamma}{4}\epsilon_{ij},$$

$$\{X_i, P_j\}_{\mathrm{D}} = \delta_{ij},\ \ \{X_i, \tilde{P}_j\}_{\mathrm{D}} = \{P_i, P_j\} = 0, \qquad (23)$$

where the relations given in (13) were used.

Note that due to the parameter γ, non-commutativity is introduced in the coordinate sector [15–17]. One can now consider the dynamics of the model using this non-commutative framework, and rewrite the Hamiltonian (10) as

$$H = \frac{1}{2m}P_i g_{ij} P_j - \frac{2m}{\gamma^2}\tilde{P}_i g_{ij}\tilde{P}_j. \qquad (24)$$

We thus see, in the non-commutative phase space, that the Hamiltonian, (10), can be diagonalized so that it describes a free motion (external modes) supplemented by the oscillator modes ("internal" modes) with negative sign of their energies [12]. But, as in [12], the variables $\tilde{P}_i$ can be identified with a standard pair of canonical variables. Indeed, identifying $\tilde{P}_1 = \sqrt{\gamma}\ \tilde{x}$, $\tilde{P}_2 = \sqrt{\gamma}\ \tilde{p}$ and introducing the oscillator variables

$$C = \frac{1}{\sqrt{2}}(\tilde{x} + \mathrm{i}\tilde{p}),\ \ C^* = \frac{1}{\sqrt{2}}(\tilde{x} - \mathrm{i}\tilde{p}), \qquad (25)$$

we find that the Hamiltonian (24) can be rewritten as

$$H = \frac{1}{2m}P_i g_{ij} P_j - \frac{2m}{\gamma}(C^2 + C^{*2}) \qquad (26)$$

and from (25) that $\{C, C^*\}_{\mathrm{D}} = -\mathrm{i}/2$.

Note that, unlike [12], in our model we do not need to impose a subsidiary condition $(C|\mathrm{phys}\rangle = 0)$ because in this case

$$\langle\mathrm{phys}|(C^2 + C^{*2})|\mathrm{phys}\rangle = 0. \qquad (27)$$

So we see that the second term in (26) do not contribute, on average, to the spectrum of the Hamiltonian (26).

5 Introducing an interaction to the free model

Next we shall introduce interactions to the free Lagrangian (5), by a potential energy term, which do not modify the internal Hamiltonian (second term in (24)) and add to the free external Hamiltonian (first term in (24)), an arbitrary potential $U(X)$ involving non-commutative variables, as follows:

$$H^{(\mathrm{ext})} = \frac{1}{2m}P_i g_{ij} P_j + U(X). \qquad (28)$$

This leads to deformation of the constraint algebra, since the secondary constraint, (11), now involves a derivative of the potential. In the simplest case one can assume that the potential U is quadratic (electric harmonic potential), so

$$H^{(\mathrm{ext})} = H_1^{(\mathrm{ext})} - H_2^{(\mathrm{ext})}, \qquad (29)$$

where

$$H_i^{(\mathrm{ext})} = \frac{1}{2m}P_i^2 + \frac{m\omega^2}{2}X_i^2,\ i = 1, 2. \qquad (30)$$

The Hamilton equations in the non-commutative space, using (23), can be written as

$$\dot{X}_i = \left\{X_i, H_j^{(\mathrm{ext})}\right\}_{\mathrm{D}} = \frac{\gamma\omega^2}{m}\epsilon_{ij}X_j - \frac{1}{m}P_i,$$
$$\dot{P}_i = \left\{P_i, H_j^{(\mathrm{ext})}\right\}_{\mathrm{D}} = -m\omega^2 X_i. \qquad (31)$$

So, we get the following equation of motion for the non-commutative variable X_i:

$$m\ddot{X}_i - \gamma\omega^2\epsilon_{ij}\dot{X}_j - m\omega^2 X_i = 0. \qquad (32)$$

Notice that the system in the space of non-commutative coordinates behaves as a set of coupled harmonic oscillators, where the coupling term is due to the radiation damping constant γ. So, when the field reaction is not considered, the system behaves like a set of decoupled harmonic oscillators. Here X is given by (21) and ω is the frequency. Introducing, in the standard way, the oscillator variables

$$A_i = \sqrt{\frac{m\omega}{2}}X_i + \mathrm{i}\sqrt{\frac{1}{2m\omega}}P_i, \qquad (33)$$

$$A_i^* = \sqrt{\frac{m\omega}{2}}X_i - \mathrm{i}\sqrt{\frac{1}{2m\omega}}P_i, \qquad (34)$$

we get

$$H_{1(2)}^{(\mathrm{ext})} = \frac{\omega}{2}\left(A_{1(2)}A_{1(2)}^* + A_{1(2)}^*A_{1(2)}\right). \qquad (35)$$

From the Dirac brackets (see (23)) for the non-commutative variables X_i and P_i, using the substitution $\{\cdot,\cdot\}_{\mathrm{D}} \to (1/\mathrm{i}\hbar)[\cdot,\cdot]$, one can find that

$$\left[A_i, A_j^\dagger\right] = \hbar\delta_{ij} - \frac{\mathrm{i}\hbar\gamma\omega}{2m}\epsilon_{ij}\,, \qquad (36)$$

$$[A_i, A_j] = \left[A_i^\dagger, A_j^\dagger\right] = -\frac{\mathrm{i}\hbar\gamma\omega}{2m}\epsilon_{ij}\,. \qquad (37)$$

The parameter γ introduces a deformation of the Heisenberg commutation relations, which does not obstruct the quantization of the model as well discussed by Banerjee et al. in [2]. However, we are interested in quantizing the model in the commutative phase space. To this end we build a commutative phase space introducing the following space coordinates:

$$\hat{X}_i = X_i - \frac{\gamma}{2m^2}\epsilon_{ij}P_j$$
$$= x_i - \frac{2}{m}g_{ij}\tilde{p}_j - \frac{\gamma}{2m^2}\epsilon_{ij}p_j, \qquad (38)$$

where

$$\left\{\hat{X}_i, P_j\right\}_{\mathrm{D}} = \delta_{ij},$$
$$\left\{\hat{X}_i, \hat{X}_j\right\}_{\mathrm{D}} = \left\{\hat{X}_i, \tilde{P}_j\right\}_{\mathrm{D}} = 0. \qquad (39)$$

Note that the commuting variables $\hat{X}_i$ are not physical, since they transform incorrectly under Galilean boosts, namely as $\{\tilde{B}_i, \hat{X}_j\} = \delta_{ij}t - \frac{\gamma}{m}\epsilon_{in}g_{nj}$. So, the coordinates X_i describing the non-commutative plane are identified uniquely as the coordinates forming a Galilean vector.

Hence we can write the Hamiltonian (29) using the variables $(\hat{X}, P)$ [14] as follows:

$$H_i^{(\mathrm{ext})} = \frac{P_i^2}{2\tilde{m}} + \frac{\tilde{m}\tilde{\omega}^2}{2}\hat{X}_i^2 + \frac{\gamma\omega^2}{2m}\epsilon_{ij}\hat{X}_iP_j, \qquad (40)$$

where $i = 1, 2$ and

$$\tilde{m} = m\left(1 + \omega^2\frac{\gamma^2}{4m^2}\right)^{-1}, \qquad (41)$$

$$\tilde{\omega}^2 = \omega^2\left(1 + \omega^2\frac{\gamma^2}{4m^2}\right). \qquad (42)$$

If we introduce the standard quantized oscillator variables

$$a_i = \sqrt{\frac{\tilde{m}\tilde{\omega}}{2\hbar}}\hat{X}_i + \mathrm{i}\sqrt{\frac{1}{2\tilde{m}\tilde{\omega}\hbar}}P_i, \qquad (43)$$

$$a_i^\dagger = \sqrt{\frac{\tilde{m}\tilde{\omega}}{2\hbar}}\hat{X}_i - \mathrm{i}\sqrt{\frac{1}{2\tilde{m}\tilde{\omega}\hbar}}P_i, \qquad (44)$$

we find that

$$H^{(\mathrm{ext})} = \hbar\tilde{\omega}(a_1^\dagger a_1 - a_2^\dagger a_2) + \mathrm{i}\hbar\frac{\omega^2\gamma}{2m}(a_1^\dagger a_2^\dagger - a_1a_2). \qquad (45)$$

Introducing the following notation:

$$\Omega = \omega\left(1 + \omega^2\frac{\gamma^2}{4m^2}\right)^{1/2}, \quad \Gamma = \omega^2\frac{\gamma}{2m}, \qquad (46)$$

Equation (45) can be rewritten as

$$\mathcal{H}^{(\mathrm{ext})} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{I}}, \qquad (47)$$

where

$$\mathcal{H}_0 = \hbar\Omega\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad (48)$$

and

$$\mathcal{H}_{\mathrm{I}} = \mathrm{i}\hbar\Gamma\left(a_1^\dagger a_2^\dagger - a_1a_2\right). \qquad (49)$$

Following [3], we see that the dynamical group structure associated with our system is that of $SU(1,1)$. The generators of this algebra are

$$J_+ = a_1^\dagger a_2^\dagger, \quad J_- = J_+^\dagger = a_1a_2, \qquad (50)$$

$$J_3 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \qquad (51)$$

corresponding to the Casimir operator $\mathcal{C} = \frac{1}{4} + J_3^2 - \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{4}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)^2$. The Hamiltonians (48) and (49) are then rewritten as

$$\mathcal{H}_0 = 2\hbar\Omega\mathcal{C}, \quad \mathcal{H}_{\mathrm{I}} = -2\hbar\Gamma J_2, \qquad (52)$$

where $[\mathcal{H}_0,\ \mathcal{H}_{\mathrm{I}}] = 0$, as $\mathcal{H}_0$ is in the center of the dynamical algebra.

Let us denote by $|n_1, n_2\rangle$ the set of simultaneous eigenvectors of $a_1^\dagger a_1$ and $a_2^\dagger a_2$, with n_1, n_2 non-negative integers. One can see that the eigenvalue of $\mathcal{H}_0$ in this frame is the constant quantity $2\hbar\Omega(n_1 - n_2)$. The eigenstates of $\mathcal{H}_\mathrm{I}$ can be written in the standard basis, in terms of the eigenstates of $\left(J_3 - \frac{1}{2}\right)$ in the representation labeled by the value $j \in Z_{1/2}$ of $\mathcal{C}$, $\{|j,m\rangle\,; m \geq x\,|j|\}$:

$$\mathcal{C}\,|j,m\rangle = j\,|j,m\rangle\,, \quad j = \frac{1}{2}(n_1 - n_2); \tag{53}$$

$$\left(J_3 - \frac{1}{2}\right)|j,m\rangle = m\,|j,m\rangle\,, \quad m = \frac{1}{2}(n_1 + n_2). \tag{54}$$

Note that this Hamiltonian is similar to the one obtained in [3] for the damped harmonic oscillator, with the difference in the notation introduced in (46). It is important to point out that, as in [3], our Hamiltonian formulation (47) is the simple undamped harmonic oscillator when $\gamma \to 0$ ($\Omega \to \omega$). The states generated by $a_2^\dagger$ represent the sink where the energy dissipated by the accelerated charge particle flows. We see therefore that the a_2-system thus represents the reservoir or heat bath coupled to the a_1-system.

6 Concluding remarks

We have shown that in the pseudo-Euclidean metrics the Lagrangian density of the system made of a charge interacting with its own radiation and of its time-reversed image, this last introduced by doubling the degrees of freedom as required by the canonical formalism, actually behaves as a closed system described by the Lagrangian (3). On the hyperbolic plane, (6) shows that the dissipative term actually acts as a coupling between the systems x_1 and x_2. Our model can be interpreted as describing a free motion in the $D = 2$ space with non-commuting coordinates and internal modes with negative energies. However, because of the pseudo-Euclidean metrics, we have shown that these internal modes do not contribute to the energy spectrum. We do not need to impose a subsidiary condition. Finally, by introducing the commuting position variables (see (38)), we observe that the quantum Hamiltonian is obtained and that the dynamical group structure associated with our system is that of $SU(1,1)$. Also, it is shown that the noncommutative coordinates X_i are identified uniquely as the coordinates forming the Galilean vector. In future works, we will study the supersymmetric extension and the introduction of gauge interactions into the model.

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References

1. M. Blasone, P. Jizba, G. Vitiello, Phys. Lett. A **287**, 205 (2001)
2. R. Banerjee, P. Mukherjee, quant-ph/0205143; quant-ph/0108055
3. E. Celechini, M. Rasetti, G. Vitiello, Ann. Phys. **215**, 156 (1992)
4. A.C.R. Mendes, F.I. Takakura, Phys. Rev. E **64**, 056501 (2001)
5. H. Feshbach, Y. Tikochinsky, Trans. N.Y. Acad. Sci., Ser. II **38**, 44 (1977)
6. W. Heitler, The quantum theory of radiation (Dover, 1970), 3nd ed.; J.D. Jackson, Classical electrodynamics (Wiley, New York 1975), 2nd ed., Chaps. 14 and 17; Gilbert N. Plass, Rev. Mod. Phys. **33**, 37 (1961)
7. R.P. Walker, Radiation Damping, Proceedings, General Accelerator Physics, CERN Geneva, CERN 91-04, 116-135 (1990)
8. F.V. Hartemann, A.K. Kerman, Phys. Rev. Lett. **76**, 624 (1996)
9. C. Bula, K.T. McDonald, E.J. Prebys, C. Bamber, S. Boege, T. Kotseroglou, A.C. Melissinos, D.D. Meyerhofer, W. Ragg, D.L. Burke, R.C. Field, G. Horton-Smith, A.C. Odian, J.E. Spencer, D. Walz, S.C. Berridge, W.M. Bugg, K. Shmakov, A.W. Weidemann, Phys. Rev. Lett. **76**, 3116 (1996)
10. A.C.R. Mendes, P.M.V.B. Barone, Monograph: Abordagens Fenomenológicas e Microscópicas da Teoria do Amortecimento Radiativo, UFJF (1995)
11. M. Blasone, E. Graziano, O.K. Pashaev, G. Vitiello, Ann. Phys. **252**, 115 (1996)
12. J. Lukierski, P.C. Stichel, W.J. Zakrzewski, Ann. Phys. **260**, 224 (1997)
13. M.S. Plyushchay, Nucl. Phys. B **362**, 54 (1991); Phys. Lett. B **262**, 71 (1991)
14. Y. Brihaye, C. Gonera, S. Giller, P. Kosiński, hep-th/9503046
15. L. Faddeev, R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988); J. Barcelos Neto, C. Wotzasek, Mod. Phys. Lett. A **7**, 1737 (1992); Int. J. Mod. Phys. A **7**, 4981 (1992)
16. R. Jackiw, in Constraints Theory and Quantization Methods, edited by F. Colomo et al. (World Scientific, Singapore 1994), p. 163
17. A. Kempf, J. Math. Phys. **35**, 4483 (1994)